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Augmented Lagrangian and Tchebycheff Approaches in Multiple Objective Programming

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Abstract. Relationships between the Tchebycheff scalarization and the augmented Lagrange multiplier technique are examined in the framework of general multiple objective programs (MOPs). It is shown that under certain conditions the Tchebycheff method can be represented as a quadratic weighted-sums scalarization of the MOP, that is, given weight values in the former, the coefficients of the latter can be found so that the same efficient point is selected. Analysis for concave and linear MOPs is included. Resulting applications in multiple criteria decision making are also discussed.

Key words: Augmented Lagrangian, Efficient solutions, Multiple objective programs, Tchebycheff scalarization

1. Introduction

A variety of scalarization methods for finding efficient solutions of multiple objective programs (MOPs) have been developed over the last two decades. Some of the methods were designed specifically for linear problems and others work well only on problems with concave objective functions and a convex feasible region. One of a few methods that can generate efficient solutions of general MOPs is the Tchebycheff scalarization that selects an efficient solution based on the minimization of the Tchebycheff distance of the objective functions from an ideal point. Among the first who proposed to apply the Tchebycheff norm to MOPs in the early seventies were Bowman [4], Yu [24] and Zeleny [25]. In the eighties, that direction of research was explored by Choo and Atkins [5], Ecker and Shoemaker [6], Kaliszewski [10], Wierzbicki [23], and many others. In particular, Steuer and Choo [18] showed that by means of the lexicographic weighted Tchebycheff program every efficient solution of the nonconcave MOP is uniquely computable and all objective function values returned by this program are nondominated.

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In parallel, the theory of generalized Lagrangian duality in single objective mathematical programming has been developed as a means for resolving a duality gap that may exist for nonconcave problems. Everett [7] was perhaps the first to propose a generalized Lagrange multiplier method. Specific generalized Lagrangian functions were introduced by Roode [15]. Gould [9] proposed a multiplier function and Nakayama et al. [13] continued in that direction by adding to the requirements of the multiplier function. When the generalized Lagrangian function is of a quadratic form, then it is referred to as augmented. Various methods utilizing augmented Lagrange multiplier techniques have also been developed. Theoretical foundations for the development of augmented Lagrange multiplier techniques were given by Rockafellar [14] who thoroughly analyzed the augmented Lagrange function and obtained global saddle point conditions for general nonconcave mathematical programs. Tind and Wolsey [22] surveyed various results of generalized duality and gave a unifying framework for handling both nonlinear and integer problems. Minoux [11] gave a comprehensive summary of the generalized duality theory.

As the generalized Lagrangian duality theory plays a major role for the analysis and solution of general constrained single objective nonlinear programs, it also turned out to be helpful in generating efficient solutions of nonconcave MOPs. TenHuisen and Wiecek proposed a framework for developing generalized-Lagrangian-type scalarizing functions for nonconcave MOPs [19]. They used augmented-Lagrangian-type scalarizing functions to generate nondominated solutions of bicriteria programs [20] and multiple criteria programs [21]. A vector-valued generalized Lagrangian was recently constructed and analyzed by Singh et al. [16].

The purpose of this article is to examine relationships between the Tchebycheff method and the augmented Lagrange multiplier technique. We show that under certain conditions the Tchebycheff scalarization can be represented as a quadratic program, formulated in the objective space of the original MOP, whose coefficients come from the dual augmented Lagrangian problem of the Tchebycheff problem. In particular, for given weight values in the Tchebycheff method we give a direct calculation of the multipliers in the quadratic program leading to the selection of the same efficient solution. We also show that when applied in multiple criteria decision making, the Tchebycheff method can be complemented with the quadratic program so that additional utility information is available to the decision maker.

In Section 2 we derive the quadratic program for the general nonconcave MOP. A detailed analysis of the bicriteria case is included in Section 3. Section 4 contains a similar development for concave as well as linear MOPs, where the Tchebycheff scalarization is related to the well known weighted-sums approach (see Geoffrion [8]) by means of the classical Lagrangian duality. Implications in multiple criteria decision making are discussed in Section 5 and conclusions are contained in Section 6.

2. The nonconcave case

In this section we shall consider mathematical programs with general nonconcave objective functions and constraint functions. We first formulate the general problem, review the Tchebycheff scalarization and then introduce the dual augmented Lagrangian problem. Under suitable conditions, the dual problem leads to a quadratic program that is, in fact, a quadratic weighted-sums scalarization of the original MOP. We analyze the relationship between the two scalarizations as far as their coefficients and the efficient solutions generated by them.

First some notation. Let $x \in \mathbb{R}^n$ and introduce the objective functions $f_i(x) : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and the constraint functions $h_j : \mathbb{R}^n \to \mathbb{R}$, where $j = 1, \ldots, p$. Let

$$f(x) = [f_1(x), \ldots, f_m(x)]^T$$

and

$$h(x) = [h_1(x), \ldots, h_p(x)]^T.$$

The set X of feasible solutions is thus given by

$$X = \{x \in \mathbb{R}^n \mid h_j(x) = 0 \text{ for } j = 1, \ldots, p\}.$$

If the original problem includes an inequality feasibility constraint $\bar{h}_i(x) \leq 0$, then this constraint can be easily converted to an equality $h_i(x) = \bar{h}_i(x) + s_i^2 = 0$, where $s_i \in R$ is an unconstrained slack variable.

Consider the general multiple objective program (MOP) of the form:

$$\begin{aligned} \max \quad & f(x) \\ \text{s.t.} \quad & x \in X. \end{aligned} \tag{1}$$

A point $x^0 \in X$ is called an efficient solution of MOP if there is no other point $x \in X$ such that $f_i(x) \geq f_i(x^0)$ for $i = 1, \ldots, m$ with strict inequality holding for at least one component. The image $f(x^0)$ of an efficient solution x^0 in the objective space is called a nondominated solution.

We shall treat (1) by the Tchebycheff approach in order to find its efficient solutions. So, introduce weights $\lambda_i \in \mathbb{R}$ for $i = 1, \ldots, m$ and let z_i^* for $i = 1, \ldots, m$ be the elements of an ideal, i.e.

$$z_i^* = \{\max_{x \in X} f_i(x) + \epsilon_i \mid \epsilon_i \geq 0\}. \tag{2}$$

It is sufficient to require $\epsilon_i > 0$, but most cases allow $\epsilon_i = 0$, see Steuer [17]. Now consider the problem

$$\min_{x \in X} \max_{1 \leq i \leq m} \{\lambda_i(z_i^* - f_i(x))\}, \tag{3}$$

where $\lambda_i \geq 0$ for $i = 1, \ldots, m$ and $\sum_{i=1}^m \lambda_i = 1$.

Unless multiple criterion vectors solve (3) for a given set of weights (in which case multiple nondominated solutions of (1) may have been generated), all efficient solutions of (1) can be found as optimal solutions of (3) by changing the λ-values. However, if the multiple optimal criterion vectors exist, an additional step may be required to ensure efficiency. This can for example be done by application of the lexicographic weighted Tchebycheff approach, see Steuer [17], which lexicographically solves the problem

$$\begin{aligned} \min \quad & [\alpha, \sum_{i}^{m}(z_i^* - f_i(x))] \\ \text{s.t.} \quad & -\lambda_i(z_i^* - f_i(x)) \le \alpha,\ i = 1, \dots, m, \\ & x \in X \end{aligned} \tag{4}$$

and yields efficient solutions.

Program (3) may be written in an alternative form as

$$\begin{aligned} \min_{\alpha, x} \quad & \alpha \\ \text{s.t.} \quad & -\lambda_i f_i(x) - \alpha + \lambda_i z_i^* \le 0, i = 1, \dots, m, \\ & x \in X, \end{aligned} \tag{5}$$

where $\alpha \in \mathbb{R}$. We assume that program (5) has an optimal solution for every $\lambda_i \ge 0$ for $i = 1, \dots, m$. Observe that α will never be negative and at least one of the inequality constraints of this program will be binding at optimality.

Assume now that $\lambda_i > 0$ for $i = 1, \dots, m$ and let $(\bar{x}, \bar{\alpha})$ denote an optimal solution of program (5) so that all the inequality constraints are binding at $(\bar{x}, \bar{\alpha})$. The corresponding nondominated point $f(\bar{x})$ has components $z_i^* - \frac{(\bar{\alpha})}{(\lambda_i)}$ for $i = 1, \dots, m$. To facilitate notation we introduce

$$g_i(x, \alpha) = -\lambda_i f_i(x) - \alpha + \lambda_i z_i^* \quad \text{for } i = 1, \dots, m$$

and

$$g(x, \alpha) = [g_1(x, \alpha), \dots, g_m(x, \alpha)]^T.$$

Hence, we restrict all the inequalities to equalities and consider the following primal problem:

$$\begin{aligned} \min \quad & \alpha \\ \text{s.t.} \quad & g(x, \alpha) = 0 \\ & x \in X. \end{aligned} \tag{6}$$

We shall treat this problem by the augmented Lagrange multiplier approach. So, let us introduce dual variables (a, y), where $a \in R, a > 0$ and $y \in \mathbb{R}^m$. Let A be the

$m \times m$ diagonal matrix with diagonal elements all equal to a. We can now define the augmented Lagrange function:

$$L_Q(x, \alpha, a, y) = \alpha + g(x, \alpha)^T A g(x, \alpha) + y^T g(x, \alpha). \tag{7}$$

and the following dual program:

$$\max_{a>0, y} \min_{x \in X, \alpha} L_Q(x, \alpha, a, y). \tag{8}$$

Following Rockafellar [14], we can state the following duality result. Subject to the conditions discussed below, program (6) has an optimal solution (x, α) if and only if its dual (8) has an optimal solution (a, y), and in this case the objective values of both programs are equal. In fact, $L_Q(x, \alpha, a, y)$ has a saddle point in the primal variables (x, α) and the dual variables (a, y). This implies that an optimal solution (x, α) of (6) can be found as an optimal solution of the inner problem of (8) keeping the dual solution (a, y) fixed at the optimal values.

The conditions require that the primal problem, that is program (6), satisfy the quadratic growth condition and be stable of degree 2. The former requires that the dual problem, that is program (8), be feasible, i.e. there exists $(a, y), a > 0$ such that $\min_{x \in X, \alpha} L_Q(x, \alpha, a, y) > -\infty$. With the former satisfied, the latter is necessary and sufficient for the strong duality to hold between the primal and the dual problem. In particular, stability of degree 2 is achieved when the primal problem satisfies the quadratic growth condition and the second order sufficiency conditions (see Bazaraa et al. [3]) with y as a vector of multipliers (of the classical Lagrangian) hold at a unique optimal solution (x, α) of the primal problem in the strong sense.

We now present our main result relating an optimal solution of the primal problem to an optimal solution of a certain quadratic program.

THEOREM 1. *Let primal problem* (6) *satisfy the quadratic growth condition and be stable of degree 2.* $(\bar{x}, \bar{\alpha})$ *is an optimal solution of* (6) *if and only if* $\bar{x}$ *is an optimal solution of the quadratic program*

$$\max_{x \in X} f(x)^T Q f(x) + p^T f(x), \tag{9}$$

where Q *is a symmetric* $m \times m$ *matrix and* $p \in \mathbb{R}^m$.

Proof. Let $(\bar{a}, \bar{y})$ denote an optimal solution of (8). We shall show that α can be eliminated from (8) and that we can obtain an optimal solution of (6) by solving (9). The equality constraints of primal problem determine the value of α as

$$\alpha = \frac{1}{m} \sum_{i=1}^{m} \lambda_i (z_i^* - f_i(x)). \tag{10}$$

As $a > 0$, $L_Q(x, \alpha, \bar{a}, \bar{y})$ is strictly convex with respect to α. So for fixed $x, \bar{a}$, and $\bar{y}$, minimization of (8) with respect to α can be done through differentiation.

Hence we calculate

$$\frac{\partial L_Q(x,\alpha,\bar{a},\bar{y})}{\partial\alpha} = 1 + \sum_{i=1}^{m}(-2\bar{a}\lambda_i z_i^* - \bar{y}_i + 2\bar{a}\alpha + 2\bar{a}\lambda_i f_i(x)) = 0. \tag{11}$$

The optimal value $\alpha(x)$ of α as a function of x is then

$$\alpha(x) = \frac{1}{m}\sum_{i=1}^{m}\lambda_i(z_i^* - f_i(x)) + \frac{1}{2\bar{a}m}\left(\sum_{i=1}^{m}\bar{y}_i - 1\right). \tag{12}$$

Expressions (10) and (12) imply that

$$\sum_{i=1}^{m}\bar{y}_i = 1. \tag{13}$$

By insertion of the appropriate terms the augmented Lagrangian (7) undertakes the following form for fixed $\bar{a}$ and $\bar{y}$:

$$\begin{aligned} L_Q(x,\alpha,\bar{a},\bar{y}) &= \alpha + \bar{a}\sum_{i=1}^{m}[\lambda_i^2(f_i(x))^2 - 2\lambda_i^2 z_i^* f_i(x) + \alpha^2 - 2\lambda_i z_i^*\alpha \\ &\quad + 2\lambda_i f_i(x)\alpha + \lambda_i^2(z_i^*)^2] + \sum_{i=1}^{m}\bar{y}_i(-\lambda_i f_i(x) - \alpha + \lambda_i z_i^*) \\ &= \bar{a}m\alpha^2 + (1 - \sum_{i=1}^{m}\bar{y}_i)\alpha - 2\bar{a}\alpha\sum_{i=1}^{m}\lambda_i(z_i^* - f_i(x)) \\ &\quad + \sum_{i=1}^{m}[\bar{a}\lambda_i^2(f_i(x))^2 - (2\bar{a}\lambda_i z_i^* + \bar{y}_i)\lambda_i f_i(x) + \lambda_i z_i^*(\bar{a}\lambda_i z_i^* + \bar{y}_i)]. \end{aligned} \tag{14}$$

Observe that the second term of (14) vanishes due to (13) whereas, using (10), the third term is equal to $-2\bar{a}m\alpha^2$. This gives us

$$\begin{aligned} L_Q(x,\alpha,\bar{a},\bar{y}) = -\bar{a}m\alpha^2 &+ \sum_{i=1}^{m}[\bar{a}(\lambda_i f_i(x))^2 + (-2\bar{a}\lambda_i z_i^* - \bar{y}_i)\lambda_i f_i(x) \\ &+ \bar{a}(\lambda_i z_i^*)^2 + \lambda_i z_i^*\bar{y}_i]. \end{aligned} \tag{15}$$

Using (10) again, we calculate

$$-\bar{a}m\alpha^2 = -\frac{\bar{a}}{m}(\lambda^T f(x))^2 + \frac{2\bar{a}}{m}\lambda^T z^*\lambda^T f(x) - \frac{\bar{a}}{m}(\lambda^T z^*)^2. \tag{16}$$

Let Λ be an $m \times m$ symmetric matrix whose (i, j)-entry is given by $\frac{(\bar{a})}{(m)}\lambda_i\lambda_j)$ for $i = 1, \ldots, m$ and $j = 1, \ldots, m$, and $\bar{\Lambda}$ be an $m \times m$ diagonal matrix with diagonal elements $\bar{a}\lambda_1^2, \ldots, \bar{a}\lambda_m^2$.

We obtain

$$\bar{a}\sum_{i=1}^{m}(\lambda_i f_i(x))^2 = f(x)^T\bar{\Lambda}f(x) \tag{17}$$

and

$$\frac{\bar{a}}{m}(\lambda^T f(x))^2 = f(x)^T\Lambda f(x). \tag{18}$$

Let τ be an m vector with elements

$$\tau_i = (2\bar{a}\lambda_i z_i^* + \bar{y}_i)\lambda_i \quad \text{for } i = 1, \dots, m. \tag{19}$$

We apply (16), (17), (18) to (15) and get

$$\begin{aligned} L_Q(x, \alpha, \bar{a}, \bar{y}) = &- f(x)^T(\Lambda - \bar{\Lambda})f(x) - (\tau^T - \frac{2}{m}\bar{a}\lambda^T z^*\lambda^T)f(x) \\ &- (z^*)^T(\Lambda - \bar{\Lambda})z^* + \sum_{i=1}^{m}\lambda_i\bar{y}_i z_i^*. \end{aligned} \tag{20}$$

We now define a matrix Q and a vector p as

$$Q = \Lambda - \bar{\Lambda}, \tag{21}$$

and

$$p = \tau - 2\Lambda^T z^*. \tag{22}$$

The second term in the right hand side of (20) is equal to

$$(\tau^T - 2(z^*)^T\Lambda)f(x),$$

which is exactly $p^T f(x)$. Defining Q as above, expression (20) becomes

$$L_Q(x, \alpha, \bar{a}, \bar{y}) = -f(x)^T Qf(x) - p^T f(x) - (z^*)^T Qz^* + \sum_{i=1}^{m}\lambda_i\bar{y}_i z^*.$$

The last two terms are constant and we are left with the first two terms to be minimized with respect to x. By changing sign in the objective we get the requested form (9). □

Our analysis until now refers to the case where all weights are positive and, at optimality, all the inequality constraints of (5) are binding. If $\lambda_k = 0$ for some $k \in \{1, \dots, m\}$ then the corresponding inequality constraint of program (5) (and also the equality constraint of program (6)) is trivially satisfied and the criterion $f_k(x)$ does not contribute to determining the optimal value of α. Similarly, if $\lambda_k = 0$ in program (9), the k-th row and the k-th column of matrix Q become zero vectors and the k-th component of vector p is also zero.

We now consider the more general case when not all of the inequality constraints of program (5) are binding at the nondominated point found.

THEOREM 2. *Let the primal problem* (5) *related to the original MOP satisfy the quadratic growth condition and be stable of degree* 2. *Let* $\bar{x}$ *be an efficient point of the MOP generated by the Tchebycheff scalarization with some ideal point* z^* *and some weights* $\lambda_i > 0$ *for* $i = 1, \ldots, m$. *The same efficient point is an optimal solution of the quadratic weighted-sums scalarization* (9), *where* Q *is a symmetric* $\bar{m} \times \bar{m}$ *matrix,* $p \in \mathbb{R}^{\bar{m}}$, *and* $\bar{m}$ *is the number of the binding inequality constraints in program* (5) *at this efficient point.*

Proof. If not all the inequality constraints of program (5) are binding at $\bar{x}$, then the criterion functions corresponding to the non-binding constraints contribute neither to determining this nondominated point nor to the trade-offs between the remaining criteria. We can drop these non-binding constraints and then formulate the quadratic program whose matrix Q and vector p will be respectively reduced. □

To complement the discussion above, we point out that if not all of the inequality constraints are binding at $\bar{x}$, we can also proceed differently than it is indicated in Theorem 2. We can find another set of weights for which these previously non-binding constraints will become binding and for these new weights derive the quadratic program of full size. For the discussion on the choice of appropriate weights see Steuer [17]. Another option is to derive the quadratic problem for a primal problem involving equality as well as inequality constraints. We would then have to use another form of the augmented Lagrangian function (see Rockafellar [14]) specially designed for inequality constraint problems. In fact, the former technique is more beneficial than the latter since all the criterion functions achieve their representation in the quadratic program.

The condition of stability of degree 2 required for the primal problem becomes significant when the two scalarizations are compared from the point of view of their effectiveness in reaching every nondominated point. As stated earlier in this section, all nondominated solutions of the original MOP can be found by means of the weighted Tchebycheff scalarization or its lexicographic version independently of the curvature of the nondominated frontier. The weighted Tchebycheff scalarization can find this nondominated solution even if it is located at a point where the nondominated frontier is not differentiable and even if additionally this solution is improperly nondominated (in the sense of Geoffrion, see [8]). However, if this scalarization generates a nondominated point in the neighborhood of which the curvature of the nondominated frontier does not allow to support the frontier by any quadratic function, then the corresponding quadratic weighted-sums scalarization cannot be constructed. In this case, in fact, the condition of stability of degree 2 does not hold. However, this situation arises only in exceptional cases.

3. The bicriteria case

We now focus our attention on the bicriteria case and analyze its quadratic weighted-sums scalarization (9) in order to better examine the structure of the related quadratic function and its relationship with the Tchebycheff scalarization. Let

$$q(f) = f^T Q f + p^T f, \tag{23}$$

where

$$f = (f_1, f_2)^T,$$

$$Q = \frac{1}{2}\bar{a}\begin{pmatrix} -\lambda_1^2 & \lambda_1\lambda_2 \\ \lambda_1\lambda_2 & -\lambda_2^2 \end{pmatrix},$$

$$p = \begin{pmatrix} \bar{a}\lambda_1^2 z_1^* - \bar{a}\lambda_1\lambda_2 z_2^* + \lambda_1\bar{y}_1 \\ \bar{a}\lambda_2^2 z_2^* - \bar{a}\lambda_1\lambda_2 z_1^* + \lambda_2\bar{y}_2 \end{pmatrix}.$$

Assume again that the nondominated point $\bar{f} = f(\bar{x})$ generated by the Tchebycheff approach for some ideal point z^* and some weights λ_1 and λ_2 makes all the inequality constraints of (5) binding. Then the ideal point and the nondominated point determine the line whose equation is

$$f_2 = \frac{\lambda_1}{\lambda_2} f_1 + z_2^* - \frac{\lambda_1}{\lambda_2} z_1^*. \tag{24}$$

In order to transform matrix Q into a diagonal form we find its eigenvalues

$$\rho_1 = 0,$$

$$\rho_2 = -\frac{1}{2}\bar{a}(\lambda_1^2 + \lambda_2^2),$$

and the corresponding (normalized) eigenvectors

$$e_1 = \frac{1}{\sqrt{\lambda_1^2 + \lambda_2^2}}\begin{pmatrix} \lambda_2 \\ \lambda_1 \end{pmatrix},$$

$$e_2 = \frac{1}{\sqrt{\lambda_1^2 + \lambda_2^2}}\begin{pmatrix} \lambda_1 \\ -\lambda_2 \end{pmatrix}.$$

Let $E = (e_1, e_2)$ be the matrix of eigenvectors and $r = (r_1, r_2)^T$ be a new variable so that

$$f = Er. \tag{25}$$

Then function $q(f)$ can now be represented as

$$q(f) = q(Er) = r^T Kr + p^T Er, \tag{26}$$

where $K = E^T QE$ is the diagonal matrix with the diagonal elements

$$k_1 = 0,$$

$$k_2 = -\frac{1}{2}\bar{a}(\lambda_1^2 + \lambda_2^2).$$

Multiplying the terms in (26) and letting

$$t_1 = \frac{(p_1\lambda_1 - p_2\lambda_2)^2}{4k_2\lambda_1\lambda_2\sqrt{\lambda_1^2 + \lambda_2^2}}$$

and

$$t_2 = -\frac{p_1\lambda_1 - p_2\lambda_2}{2k_2\sqrt{\lambda_1^2 + \lambda_2^2}},$$

we get $q(f)$ transformed to a function $\tilde{q}(r)$ of the form

$$\tilde{q}(r) = k_2(r_2 - t_2)^2 + \frac{\lambda_1\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}}(r_1 - t_1), \tag{27}$$

which represents the original function (23) in the new system of coordinates (r_1, r_2) obtained by rotating the original system (f_1, f_2). Defining a new variable $s = (s_1, s_2)^T$, where

$$\begin{aligned} s_1 &= r_1 - t_1 \\ s_2 &= r_2 - t_2, \end{aligned} \tag{28}$$

we eventually obtain a function $\bar{q}(s)$ given by

$$\bar{q}(s) = k_2 s_2^2 + \frac{\lambda_1\lambda_2}{\sqrt{\lambda_1^2 + \lambda_2^2}} s_1. \tag{29}$$

Equation (29) represents the original function (23) in the new system of coordinates (s_1, s_2) obtained by rotating and translating the original system (f_1, f_2). The rotation angle γ is such that

$$\tan\gamma = \frac{\lambda_1}{\lambda_2} \tag{30}$$

and the translation vector is given by $(t_1, t_2)^T$. We observe that the weights are solely responsible for the rotation whose angle agrees with the slope of line (24) while a combination of the weights, the ideal point coordinates and the optimal values of the dual variables determine the translation. Furthermore, the s_1-axis considered in the original system (f_1, f_2) coincides with line (24).

We now calculate $f_2(s_1)$, the f_2-intercept of the s_1-axis. From (25) and (28) we obtain $E(s + t) = f$, which for $f = (0, f_2(s_1))$ yields

$$E\begin{pmatrix} s_1 + t_1 \\ s_2 + t_2 \end{pmatrix} = \begin{pmatrix} 0 \\ f_2(s_1) \end{pmatrix}. \tag{31}$$

Taking $s_2 = 0$, from (31) we obtain

$$f_2(s_1) = z_2^* - \frac{\lambda_1}{\lambda_2} z_1^* + \frac{-\lambda_1^2 \bar{y}_1 + \lambda_2^2 \bar{y}_2}{\bar{a}\lambda_2(\lambda_1^2 + \lambda_2^2)}. \tag{32}$$

As this intercept is equal to the intercept of (24), we have $\lambda_1^2 \bar{y}_1 - \lambda_2^2 \bar{y}_2 = 0$ or

$$\lambda_i^2 \bar{y}_i = \text{const}, \quad i = 1, 2. \tag{33}$$

Using (13) we find the optimal values of the dual variables

$$(\bar{y}_1, \bar{y}_2) = \left(\frac{\lambda_2^2}{\lambda_1^2 + \lambda_2^2}, \frac{\lambda_1^2}{\lambda_1^2 + \lambda_2^2} \right). \tag{34}$$

4. The concave case

In this section we return to the multiple objective case and additionally assume that all the objective functions are concave, the feasible set is convex, and that an appropriate constraint qualification holds so that the strong duality theorem for single objective concave nonlinear programs holds (see Bazaraa et al. [3]). Consider then the concave MOP of form (1), its Tchebycheff scalarization (3), and problem (5) now referred to as the primal problem to be treated by the Lagrange multiplier approach. We introduce dual variables $y \in \mathbb{R}^m$, $y \geq 0$, and define the Lagrange function:

$$L(x, \alpha, y) = \alpha + y^T g(x, \alpha), \tag{35}$$

and the following dual program:

$$\max_{y \geq 0} \min_{x \in X, \alpha} L(x, \alpha, y). \tag{36}$$

THEOREM 3. *Let the criterion functions f_i for $i = 1, \ldots , m$ be concave and the feasible set X be convex. Let the primal problem* (5) *satisfy a constraint qualification so that the strong duality theorem for single objective concave nonlinear* *programs holds. A point $\bar{x}$ is an optimal solution of* (5) *if and only if it is an optimal solution of the weighted-sums scalarization of the MOP*

$$\max_{x\in X} \mu^T f(x), \tag{37}$$

where $\mu \in R^m$ and $\mu \geq 0$.

Proof. Let $\bar{y}$ be an optimal solution of (36). For fixed x and $\bar{y}$, minimization of (36) with respect to α can again be done through differentiation. Hence from

$$\frac{\partial L(x, \alpha, \bar{y})}{\partial \alpha} = 0,$$

we get

$$\sum_{i=1}^{m} \bar{y}_i = 1.$$

Therefore the optimal value of the inner problem in (36) is

$$\min_{\alpha} L(x, \alpha, \bar{y}) = -\sum_{i=1}^{m} \lambda_i \bar{y}_i f_i(x) + \sum_{i=1}^{m} \lambda_i \bar{y}_i z_i^*. \tag{38}$$

Letting

$$\mu_i = \lambda_i \bar{y}_i \quad \text{for } i = 1, \ldots, m, \tag{39}$$

and

$$c_0 = \sum_{i=1}^{m} \lambda_i \bar{y}_i z_i^*,$$

we get

$$\min_{\alpha} L(x, \alpha, \bar{y}) = -\mu^T f(x) + c_0. \tag{40}$$

and dual problem (36) is equivalent to (37). □

Note that when $\mu > 0$, program (37) finds properly efficient solutions (in the sense of Geoffrion, see [8]). Due to (39), we must have that $\lambda > 0$ and $\bar{y} > 0$. The former agrees with Steuer [17, Section 14.9, Step 11] in the case that the ideal is selected such that $\epsilon_i > 0$ in (2) while the latter implies that all the inequality constraints of (5) are binding and nondegenerate at this solution.

The concave case implies a similar result for the linear multiple objective program (LMOP) of the form:

$$\begin{array}{rl} \max & Cx \\ \text{s.t.} & x \in S, \end{array} \tag{41}$$

where C is an $m \times n$ matrix and S is a convex polyhedral set in R^n.

THEOREM 4. *A point $\bar{x} \in S$ is an optimal solution of the weighted Tchebycheff scalarization of the LMOP if and only if it is an optimal solution of its weighted sums scalarization*

$$\max_{x \in S} \mu^T Cx, \tag{42}$$

where $\mu \in R^m$ and $\mu \geq 0$.

Proof. The proof uses the strong duality theorem of linear programming (see Bazaraa et al. [2]) and follows the derivation of the concave case. □

5. Significance in decision making

For general nonconcave MOPs, the weighted Tchebycheff scalarization (3) and the quadratic weighted-sum scalarization (9), when considered as a tool for finding efficient solutions, become related mathematical procedures since they generate the same solution using appropriate values of their input parameters. This connection grounded in theory of mathematical programming does not just carry over to multiple criteria decision making but delivers new information for conducting a decision making process, which we now discuss.

It can be shown that the matrix Q of program (9) is negative semi-definite so that this program involves maximization of a concave quadratic function of the objective functions over the image of the feasible set in the objective space. As a result, the decision maker's utility represented by the weighted Tchebycheff metric achieves a new representation by means of a concave quadratic function. Due to concavity, the unconstrained maximum value of this quadratic function is unbounded, which shows that the new utility function offers infinite improvement of all the objective function values. This implies that moving the ideal point defined for any nonnegative $\epsilon_i, i = 1, \ldots, m$, may also infinitely improve the objective values. We emphasize that this quadratic utility function should be more attractive to the decision maker than the function represented by the level curves of the weighted Tchebycheff metric. In fact, the Tchebycheff metric offers an unrealistic utility function with zero or infinite trade-offs where weakly nondominated solutions and nondominated solutions are considered to be of equal utility to the decision maker. On the contrary, in the neighborhood of the nondominated point found, the quadratic function provides the decision maker with finite trade-offs and assigns equal utility only to nondominated solutions.

As quadratic program (9) is composed of quadratic weighted sums of the objective functions, it may become computationally complex. On the other hand, Tchebycheff program (5) is easy to solve. As the two programs generate the same efficient solution, the latter can work as a tool for finding this solution while the former can specify a decision maker's utility function in the neighborhood of this solution. In this way both programs complement each other and assist a decision maker in the decision making process. Furthermore, constructing the utility function does not require any additional information that is needed to perform the Tchebycheff scalarization and find a nondominated solution. Therefore, a decision making process of searching for the most preferred efficient solutions can be carried with the Tchebycheff scalarization and additional utility information can be concurrently extracted from the related quadratic weighted-sums scalarization.

This interpretation of the quadratic program (9) follows upon the recent developments on relating utility function optimization and compromise programming (see Zeleny [25]) of which the weighted Tchebycheff approach is a special case. It is a very well known fact that in real life it is almost impossible to obtain a reliable mathematical representation of decision maker's actual utility. Ballestro and Romero [1] recognized that compromise programming does not seek to determine this utility but rather seeks to determine a portion of the nondominated set where the tangency with utility function level curves likely occurs. They considered bicriteria problems and established a necessary and sufficient condition for a utility function so that its maximum is in the portion of the nondominated set found by compromise programming. The condition requires that

$$\left.\frac{\frac{\partial U(f_1,f_2)}{\partial f_i}}{\frac{\partial U(f_1,f_2)}{\partial f_j}}\right|_{f^*} = \frac{\lambda_i}{\lambda_j}, \quad i, j = 1, 2, \quad i \neq j, \tag{43}$$

where $U(f_1, f_2)$ is a scalar utility function $U : R^2 \longrightarrow R$ and $f^* = (f_1^*, f_2^*)$ is the nondominated point at which this function achieves the maximum. Later, Moron et al. [12] found general families of functions satisfying this condition.

It can be shown that the quadratic function (23) derived for the bicriteria case with the optimal values of the dual variables (34) satisfies condition (43) at the nondominated point $\bar{f}$. Clearly, the utility function given by the weighted-sum of the criteria (37) also satisfies this condition.

6. Conclusions

In this paper, a Tchebycheff-related dual approach to general nonconcave multiple objective programs (MOPs) is developed. The approach results from the application of the augmented Lagrange multiplier technique to the weighted Tchebycheff scalarization and involves optimization of quadratic weighted-sums of the original objective functions. Under suitable conditions, the Tchebycheff method and its dual approach generate the same efficient solution. The bicriteria case is studied in more detail and it is shown the weights uniquely determine the optimal values of the dual variables associated with the linear term of the augmented Lagrangian. The analysis includes concave and linear MOPs as a special case for which the optimal values of the dual variables are also derived.

The duality relationships revealed in this paper should become significant in multiple criteria decision making where the decision maker's utility represented by the weighted Tchebycheff metric can now be substituted by a concave quadratic function. Although the importance of the new function has been discussed, the role of the dual variables of the augmented Lagrange function in the decision making process should be still investigated from the point of view of providing the decision maker with more information on how to choose the most preferred efficient solution.

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References

1. Ballestro, E. and Romero, C. (1991), A theorem connecting utility function optimization and compromise programming, *Operations Research Letters* 10: 421–427.
2. Bazaraa, M.S., Jarvis, J.J. and Sherali, H.D. (1990), *Linear Programming and Network Flows*, Wiley, New York.
3. Bazaraa, M.S., Sherali, H.D. and Shetty C.M. (1993), *Nonlinear Programming – Theory and Algorithms*, Wiley, New York.
4. Bowman, V.J. (1976), On the relationship of the Tchebycheff norm and the efficient frontier of multiple-criteria objectives, *Lecture Notes in Economics and Mathematical Systems* 130: 76–85.
5. Choo, E.-U. and Atkins, D.R. (1980), An interactive algorithm for multicriteria programming, *Computers and Operations Research* 7: 81–87.
6. Ecker, J.G. and Shoemaker, N.E. (1981), Selecting subsets from the set of nondominated vectors in multiple objective linear programming, *SIAM Journal on Control and Optimization* 19: 505–515.
7. Everett, H. (1963), Generalized Lagrange multiplier method for solving problems of optimum allocation of resources, *Operations Research* 11: 399–417.
8. Geoffrion, A.M. (1968), Proper efficiency and the theory of vector maximization, *Journal of Mathematical Analysis and Applications* 22: 618–630.
9. Gould, F.J. (1969), Extensions of Lagrange multipliers in nonlinear programming, *SIAM Journal of Applied Mathematics* 17: 1280–1297.
10. Kaliszewski, I. (1987), A modified weighted Tchebycheff metric for multiple objective programming, *Computers and Operations Research* 14: 315–323.
11. Minoux, M. (1986), *Mathematical Programming: Theory and Algorithms*, Wiley, Chichester.
12. Moron, M.A., Romero, C., and Ruiz del Portal, F. (1996), Generating well-behaved utility functions for compromise programming, *Journal of Optimization Theory and Applications* 91: 643–649.
13. Nakayama, H., Sayama, H. and Sawaragi, Y. (1975), A generalized Lagrangian function and multiplier method, *Journal of Optimization Theory and Applications* 17: 211–227.
14. Rockafellar, R.T. (1974), Augmented Lagrange multiplier functions and duality in nonconvex programming, *SIAM Journal on Control* 12: 268–285.
15. Roode, J.D. (1968), *Generalized Lagrangian Functions in Mathematical Programming*, Ph.D. Thesis, University of Leiden, The Netherlands.
16. Singh, C., Bhatia, D. and Rueda, N. (1996), Duality in nonlinear multiobjective programming using augmented Lagrangian functions, *Journal of Optimization Theory and Applications* 88: 659–670.

17. Steuer R.E. (1986), *Multiple Criteria Optimization: Theory, Computation and Application*, Wiley, New York.
18. Steuer, R.E. and Choo, E.-U. (1983), An interactive weighted Tchebycheff procedure for multiple objective programming, *Mathematical Programming* 26: 326–344.
19. TenHuisen, M.L. and Wiecek, M.M. (1994), Vector optimization and generalized Lagrangian duality, *Annals of Operations Research* 51: 15–32.
20. TenHuisen, M.L. and Wiecek, M.M. (1997), Efficiency and solution approaches to bicriteria nonconvex programs, *Journal of Global Optimization* 11: 225–251.
21. TenHuisen, M.L. and Wiecek, M.M., (1997) An augmented Lagrangian scalarization for multiple objective programming, in: R. Caballero, F. Ruiz, R.E. Steuer (eds.), *Advances in Multiple Objective and Goal Programming*, pp. 151–159, Springer, Berlin.
22. Tind, J. and Wolsey, L.A. (1981), An elementary survey of general duality theory in mathematical programming, *Mathematical Programming* 21: 241–261.
23. Wierzbicki, A.P. (1980), The use of reference objectives in multiobjective optimization, *Lecture Notes in Economics and Mathematical Systems* 177: 468–486.
24. Yu, P.L. (1973), A class of solutions for group decision problems, *Management Science* 19: 262–301.
25. Zeleny, M.(1973), Compromise programming, in: J.L. Cochrane and M. Zeleny, (eds.), *Multiple Criteria Decision Making*, pp. 262–301, University of South Carolina Press, Columbia, S.C.